Calculus Warm Up # 5-4

Find the 3rd Taylor Polynomial for $f(x) = \sin x$ expanded about $c = \frac{\pi}{6}$

Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c, then for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

$x < z < c$

or $c < z < x$

$R_n(x)$ is the Lagrange form of the remainder.

Given: $f(x) = e^x$

1) Find the 2nd order Taylor Polynomial centered at c = 0 and use it to approximate f(0.2).

2) Find the Lagrange form of the remainder, $R_2(x)$ and use it to estimate the maximum error in your approximation.

1) $f'(x) = e^x$ $\quad c = 0$

$e^0 = 1$

$f''(x) = e^x$ $\quad 1$

$P_2(x) = 1 + x + \frac{x^2}{2}$

$$f(0.2) \approx P_2(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2}$$
$$= 1.22$$

Given: $f(x) = e^x$

2) Find the Lagrange form of the remainder, $R_2(x)$ and use it to estimate the maximum error in your approximation.

$P_2(x) = 1 + x + \frac{x^2}{2}$ $\qquad f(0.2) \approx 1.22$

$$R_2(x) = \frac{f'''(z)\,x^3}{3!} = \frac{e^z x^3}{6}$$

$c = 0$ $\qquad x = 0.2$

$0 < z < 0.2$

Since e^x is increasing
$e^z < e^{0.2}$

So, $\frac{e^z x^3}{6} < \frac{e^{0.2} x^3}{6}$

$$\text{error} < \frac{e^{0.2}(0.2)^3}{6}$$

Max error

$$\text{error} < 0.0016$$

Use Taylor's Th. to approximate f(0.1) and determine the accuracy of the approximation.

Given: $f(x) = \sin x$

$c=0$ $\quad x=0.1$

$0 < z < 0.1$

$P_3(x) = x - \frac{x^3}{3!}$

$\sin z < \sin(0.1)$

b/c $\sin x$ is increasing

$f(0.1) \approx P_3(0.1) = 0.1 - \frac{(0.1)^3}{6}$

$f(0.1) \approx 0.0998$

$\frac{\sin(z)(0.1)^4}{4!} < \underbrace{\frac{[\sin(0.1)](0.1)^4}{4!}}_{\text{Max error}}$

$R_3(x) = \frac{f^{(4)}(z)x^4}{4!}$

error < 0.000004

HW # 31) Determine the values of x where the error in approximation will be < 0.001.

Given: $f(x) = e^x$ $\quad P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $x<0$

$R_3(x) = \frac{f^{(4)}(z)x^4}{4!}$

$= \frac{e^z x^4}{4!}$

Since: $x<0$ and $c=0$

$x < z < 0$

$\frac{e^z x^4}{4!} < \underbrace{\frac{e^0 x^4}{4!}}_{\text{Max error}}$

$\frac{x^4}{4!} < 0.001$

Solve for x

HW # 31) Determine the values of x where the error in approximation will be < 0.001.

Given: $f(x) = e^x$ $\quad P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $x<0$

$$\frac{x^4}{4!} < 0.001$$

$$x^4 < 0.024$$

$$|x| < \sqrt[4]{0.024}$$

$$\boxed{-0.3936 < x < 0}$$

Now try MC - B # 87

$$R_3(x) = \frac{f^{(4)}(z)x^4}{4!}$$

$$R_3(x) = \frac{e^{\sin z}x^4}{4!}$$

sin x increasing on [0,1] so $e^{\sin x}$ is also increasing

$$0 < z < 1$$

$$e^{\sin z} < e^{\sin 1}$$

$$\text{error} < \underbrace{\frac{e^{\sin 1}(1)^4}{4!}}_{\text{Max error on } [0,1]}$$

Group Quiz tomorrow:

Testing for Convergence/Divergence

Writing a Taylor Polynomial for a function and using it to find an approximation.

HW: Finish MC - B (except #78)

Review notes for group quiz

Classwork turn in:

WU & MC-B

HW turn in:

MC-A, except # 7, 13, 17, 23, 26

MC part B answers:

76. B
77. B
78. D
79. E
80. B
81. A
82. C
83. D
84. C
85. B
86. E
87. B
88. C
89. B
90. D
91. B
92. C

* Still need to learn:

#78, ~~86~~, ~~87~~, ~~90~~

79) A) T → differentiable implies continuous so limit will match $f(3) = 8$

B) T ↗

C) T → alternate form of the derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

D) T → limit definition of the derivative

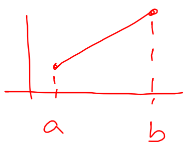
E) $\lim_{x \to 3} f'(x) = 5$ ← doesn't have to.

80) $2f'(x)$ will not change basic characteristics of h' (Increasing/Decr, +/−)

PI same as f, where f' goes from increasing to decreasing or decreasing to increasing

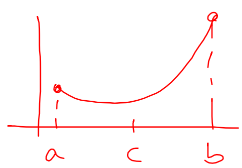
82) counterexamples (where the choice could be false)

A)



for c on (a, b)
$f(c) \neq 0$

B)



$f(c)$ is not btwn $f(a)$ and $f(b)$

D) Same as

$\boxed{C}$ f continuous,
c on $[a, b]$
$f(c)$ is on $f(x)$

E) Mean Value Theorem

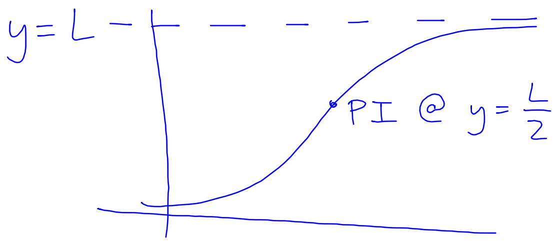
84) logistic differential equation from 9.5

$$\frac{dy}{dt} = ky(L-y)$$

L = upper growth Limit

k = proportionality constant

logistic growth curve



86) Notice alternate signs, so $f^{(30)}(3)$ will be positive.

for $n=4$ compare $\frac{f^{(4)}(3)(x-3)^4}{4!}$ to $-\frac{(x-3)^4}{2!}$

$$\frac{f^{(4)}(3)(x-3)^4}{4\cdot 3\cdot 2!}$$

must $= -1$

so $f^{(4)}(3) = -12$

$= -\frac{4!}{2!}$

sooo...

$f^{(30)}(3)$

$= \frac{30!}{15!}$

90) $\dfrac{f^{(n)}(0)\,x^n}{n!}$

check for $n=2$

$f''(0)\,\dfrac{x^2}{2!}$

$\dfrac{(-1)^2(2+1)}{(2+2)\,2^2}\cdot\dfrac{x^2}{2!}$

$= \dfrac{3}{32}x^2$

91) Inverses have reciprocal slopes

on f @ $(1,3) \longrightarrow f'(1) = 4$

on g @ $(3,1) \longrightarrow g'(3) = \dfrac{1}{4}$